**Green Growth Knowledge Platform (GGKP)**
Third Annual Conference
Fiscal Policies and the Green Economy Transition: Generating Knowledge – Creating Impact
29-30 January, 2015
Ca' Foscari University of Venice, Venice, Italy

# Energy transition under irreversibility: a two-sector approach

Prudence Dato (University of Savoie)

**The GGKP's Third Annual Conference is hosted in partnership with the Ca' Foscari University of Venice, The Energy and Resources Institute (TERI) and the United Nations Environment Programme (UNEP).**

GREEN GROWTH
Knowledge Platform

# Energy transition under irreversibility: a two-sector approach

Prudence Dato*

December 9, 2014

**Abstract**

In the present paper, we analyze the optimal energy transition of a two-sector economy (Energy and Final goods) with exhaustible oil reserves, a renewable source of energy and a pollution threat. The latter corresponds to a pollution threshold above which a part of capital is lost (following flooding for instance). Part of the energy is used as energy services by a representative consumer through a CRRA utility function and the other part is used as input in a Leontief production function to produce final goods. Moreover, we assume that both energy sources are complementary. We use the optimality conditions as in Boucekkine et al. (2013) to show that the optimal energy transition path may correspond to a corner regime in which the economy starts using both resources, then crosses the pollution threshold and therefore loses a part of its capital. At the end, the economy never adopts only renewable energy. This result goes in line with the asymptotic energy transition arguments stating that the transition to a clean energy may happen only in the long run. We extend the present model to allow for additional investment in energy savings technologies. Results mainly show that this additional investment favours the energy transition in the sense that it increases the time at which the economy may experience the catastrophe and the welfare of the society. For policy implications, economic instruments such as taxes on the "dirty" energy, subsidies on the "clean" energy or incentives for energy saving technologies need to be implemented in order to promote the energy transition. But those economic instruments should be carefully designed in line with the asymptotic energy transition result.

**Keywords:** *energy, pollution, irreversibility, switch.*

*IREGE, University of Savoie; 4, Chemin de Bellevue, BP 80439, 74944, Annecy-le-vieux CEDEX (France); Tel.: +33(0)4-50-09-24-40; fax: +33(0)4-50-09-24-39. E-mail address: prudence.dato@univ-savoie.fr.

# 1 Introduction

In order to reduce the global CO2 emissions up to 50 per cent from 2005 to 2050, several energy policies are using scenarios that mostly include the adoption of renewable energy (RE) sources and investments in energy savings technologies (EST). Despite the growing investments in the production of RE (63 to 244 billion USD from 2006 to 2012 (GEA,2012)), fossil fuels as dirty energy are still mainly used (78.2%) in the world. Therefore it becomes crucial not only to drastically change the way energy is produced, but also to look for energy saving strategies. According to the Global Energy Assessment (GEA)-Efficiency pathways, about one-third of overall investment in energy sector is efficiency related (GEA, 2012). The present paper focuses on the issue of energy transition that involves both decision of RE adoption and that of the investment in EST. We mainly analyze the optimal energy transition of a two-sector economy (energy and final goods) with exhaustible oil reserves, a renewable source of energy and a pollution threat.

The issue of energy transition involves both decision of RE adoption and that of the investment in EST. The former decision concerns an adoption of clean energy sources as an alternative that could replace the consumption of the polluting sources of energy while the latter could help reduce the overall energy consumption. In line with the economic literature, at the beginning many authors focused differently on the long run depletion of oil reserves and on the polluting fearture of oil. Dasgupta and Heal (1974, 1979), Dasgupta and Stiglitz (1981) and Krautkraemer (1986) analyze the long run depletion of oil reserves while Nordhaus (1994) and Tahvonen (1996, 1997) focus on the polluting aspects of oil. In this regard, one solution could be to adopt a backstop technology (a renewable resource for instance) as a clean energy. More recently, several works (Acemoglu et al., 2014; Amigues et al., 2013 and Tsur and Zemel, 2003) focus on climate change issues as one of the important reasons that urge the transition to a clean energy or to clean technologies. As the use of the polluting energy resource generates a pollution that accumulates over years, the ecological catastrophe may occur at some point in time. The catastrophic event generates some damages that can be irreversible[1] (Forster, 1975; Tavhonen and Withagen, 1996; Ulph and Ulph, 1997; Pindyck, 2002; Pommeret and Prieur, 2009 and Ayong Le Kama et al., 2011). It can also be partly reversible (Tsur and Zemel, 1996 and Naevdal, 2006) or fully reversible (Kollenbach, 2013).

There is no consensus in the literature about how to model the environmental damages due to pollution. Some authors consider the damages as income loss (Karp and Tsur, 2011 and Tsur and Withagen, 2012) or social welfare loss (Van der ploeg and Withagen, 2012 and Prieur et al., 2013). Differently, some other authors focus on productive sectors: capital loss (Horii and Ikefuji, 2010) or destruction capacity (Golosov et al., 2011). The present paper assumes that the economy experiences a catastrophic event (flooding for instance) when the level of pollution is above a certain critical thresh-

[1] There are various types of irreversibility. It can be an exhaustion of the natural capacity of regeneration (Tsur and Withagen, 2013), an irreversibility in the decision process (Pommeret and Prieur, 2009 and Ayong Le Kama et al., 2011) or a ceiling on the pollution stock (Lafforgue et al., 2008 and Chakravorty et al., 2012)

old. Therefore, the economy loses a part of its productive stock of capital. Moreover, to support the simultaneous use of both resources, many authors assume a convexity of the production cost of the renewable energy (Chakravorty et al., 1997; Amigues et al., 2013) or an increasing extraction costs of fossil fuels (Tsur and Zemel, 2005 and Kollenbach, 2013). For instance, Amigues et al. (2013) study energy transition in a deterministic framework and consider adjustment costs over production capacity of renewable energy. They identify three energy regimes in a partial equilibrium setting with an intermediate regime of simultaneous use of both resources. In addition, several studies assume imperfect or perfect substitution between inputs. On the opposite, we consider the case of an economy with such rigidities that oil and RE source are complementary as in Pelli (2012). Moreover, we also assume that capital use and energy are complementary, as in Pindyck and Rotemberg (1983), Boucekkine and Pommeret (2004) or Diaz and Puch (2013).

In a deterministic framework, Boucekkine et al. (2013) provide first order optimality conditions in an optimal regime switching problem with threshold effects. These optimality conditions are the continuity of appropriate co-states and states variables and that of the Hamiltonian. The present paper is mainly related to the application in Boucekkine et al.(2013) as it involves together the switching decision to a cleaner energy sources and the pollution threshold effect as the main drivers of energy transition. However, the contribution of our paper is threefold. First, we use a two-sector approach in which the economy requires capital to produce energy that can be used as inputs to produce a final good. We do not allow a natural regeneration capacity, instead we consider the irreversibility of the pollution for a loss of capital. In the same vein, we do not account for a direct pollution damage, but only the loss of the productive capital due to the occurrence of the catastrophe. In contrast to Boucekkine et al.(2013), we allow a simultaneous use of both resources (dirty and clean energies). More precisely, we assume that there is a complementarity between both resources use and also with the capital in the production of final goods. Second, we solve the model backward by using the analytical first order optimality conditions. Numerical results show that the optimal energy transition path may correspond to a corner regime in which the economy starts using both resources, crosses the pollution threshold by losing a part of its capital and never adopts only renewable energy. This result goes in line with the asymptotic energy transition argument stating that the transition to a clean energy may happen only in the long run. Sensitivity analysis shows that the optimal time to cross the pollution threshold positively depends on (i) the corresponding capital loss; (ii) the productivity of capital and energy services and (iii) the level of the pollution threshold. Moreover, it negatively depends on the discount rate.

Third, we extend our model to the adoption of energy saving technology. Very few works deal with the adoption of energy savings technologies ( Charlier et al., 2011; De Groot et al., 2001 and Acemoglu et al., 2012). In order to fill this gap in literature about the importance of EST in the energy transition, we extend our model to allow investment decision in EST. More precisely, the economy may decide to invest in energy saving appliances or in energy efficient systems to reduce the overall energy consumption.

This investment is additional to that in clean energy to help reach the energy transition targets. Numerical results mainly show that investments in energy saving technologies may help to reduce the consumption of energy for the same quality of energy services and therefore favour the energy transition. The remainder of this paper is structured as follows. The model is presented in section 2. We analyze the optimal energy transition path in section 3. Section 4 extends the model to allow investment in EST. Finally, we conclude in section 5.

# 2 Model

We consider a closed economy that produces energy and final goods in a general equilibrium setting. The economy uses a "dirty" source (exhaustible oil reserves) and a "clean" source (solar panels) to produce energy. Part of the energy is used as energy services by a representative consumer through a CRRA utility function. The other part is used as input in a Leontief production function to produce final goods. The use of dirty energy by both final goods sector and households has a negative impact on environment. Above a certain pollution threshold, the economy experiences a catastrophic event (following flooding for instance) and loses a part of its stock of capital. In the following sections, we describe the energy sector, the final good sector, households' utility and pollution threat respectively.

## 2.1 Energy sector

Energy is an intermediate good that is produced using $E_s$, a non-renewable and dirty source and $E_x$, a renewable and clean source. A representative consumer uses a part $E_2$ of the energy as energy services while the other part $E_1$ is used as input to produce final goods. Let us denote respectively $E_{2s}$, $E_{2x}$, $E_{1s}$ and $E_{1x}$ the parts of the NRE and the RE that households use and that the final goods sector uses. We assume that the production of the NRE is costless. The stock $S_t$ of the NRE at each time $t$ is generated by the following dynamics:

$$dS_t = -E_{st}dt \tag{1}$$

where $E_{st}$ is the rate of extraction of the NRE.

The production of RE requires the use of capital. For instance, to produce solar (respectively wind) energy, one needs to install some solar panels (respectively wind turbines) in order to transform solar (respectively wind) into electricity. Hence we assume a "$\varphi$-to-one" transformation of $K_1$, a part $\phi$ of capital K as follows:

$$E_x = \varphi K_1 = \varphi\phi K \tag{2}$$

where $\varphi$ is the productivity of capital in the RE sector and is greater than one ($\varphi > 1$).

In our model, pollution only comes from the use of the dirty energy. The following energy market clearing conditions holds.

The NRE that the economy produces is fully used as comsumption by households and as input to produce final goods as well:

$$E_{st} = E_{1st} + E_{2st} \tag{3}$$

The total production of the RE is split into the final goods sector and the household energy consumption:

$$E_{xt} = E_{1xt} + E_{2xt} \tag{4}$$

Finally, the total energy that is used in the economy is that from the NRE and the RE.

$$E_{1t} + E_{2t} = E_{xt} + E_{st} \tag{5}$$

## 2.2 Pollution threat

The use of the NRE either as comsumption by household or as inputs to produce final goods generates GHG emissions. Pollution accumulates in the environment (atmosphere) according to the following process:

$$\dot{Z}_t = E_{st} \tag{6}$$

We do not account for the natural regeneration capacity of environment as in Van der Ploeg and Withagen (2012, 2014). This can be seen as the most pessimistic way to deal with the threat of pollution to justify the necessity of an energy transition. Moreover, the economy experiences a catastrophic event (flooding for instance). When the level of pollution $Z_t$ is above a certain critical threshold $\overline{Z}$, the economy loses a part $\theta$ of its capital stock.

## 2.3 Final good sector

In order to produce a final good Y, a part $E_1 = E_{1st} + E_{1xt}$ of energy and a part $1 - \phi$ of capital ($K_2$) serve as inputs in a Leontief production function. Interpretation runs as follows. There exist operating costs whose size depends on the energy requirement of the capital or to any capital use corresponds a given energy requirement. Such a complementarity is assumed in order to be consistent with several studies arguing that capital and energy are complements (see for instance Berndt and Wood, 1974, Pindyck and Rotemberg, 1983, or more recently Diaz and Puch, 2013). The production function is defined as:

$$Y = min\{\alpha_2 K_2, \beta_2 E_{1t}\} \tag{7}$$

With $K_2 = (1 - \phi)K$.

Additionally, we assume that both oil and the renewable resource use are complementary. Two types of justifications can be provided. First, Pelli (2012) proves using an econometric approach that there exists some complementarity between the dirty sources of energy (oil, coal, gases) and the clean ones (hydroelectric, biomass -wood and waste-, geothermal, solar/photovoltaic, wind and nuclear). The intuition is that the production of energy using RE source, for instance through solar panels, requires oil to build the solar panels. Second, the presence of rigidities in a macroeconomic view may explain this complementary between the oil and RE source as well: it is not that easy to substitute between oil and electricity provided by solar panels. We define $E_{1t}$ as:

$$E_{1t} = min\{\frac{1}{\xi}E_{1st}, E_{1xt}\} \tag{8}$$

where $\xi$ is the part of the NRE use in the energy mix.

## 2.4 Households

We consider a representative household who uses the energy services $E_2$ and consumes a non-energy good C. The utility U represents the consumer's preferences that are expressed by the expected discounted sum of instantaneous CRRA utility flows :

$$U = \int_0^\infty u(C_t, E_{2t})e^{-\rho t}dt \tag{9}$$

and

$$u = \frac{C_t^{1-\delta}}{1-\delta} + \frac{E_{2t}^{1-\delta}}{1-\delta} \tag{10}$$

where $\rho$ is the discount rate and $\delta$ is the coefficient of relative risk aversion that is different from 1.

Both RE and NRE are complementary for the same reasons as in the final goods sector.

$$E_{2t} = min\{\frac{1}{\xi}E_{2st}, E_{2xt}\} \tag{11}$$

where $\xi$ is the part of the NRE use in the energy mix.

Households own firms in both energy and final goods sectors. They consume a part of the final good production and invest the rest to produce clean energy and final goods.

$$Y_t = C_t + \dot{K}_{1t} + \dot{K}_{2t} \tag{12}$$

In the following sections, we first analyze the optimal energy transition path. In section 4, we provide the numerical results. Finally, we extend the model to the adoption of energy savings technologies in section 5.

# 3 Optimal energy transition path

In this section, we first analyze the general energy transition path that includes the energy regime switch and the occurrence of a catastrophic event. Three regimes can occur that correspond to the energy transition path. In the first one, energy is produced by both oil and the renewable resource that are complementary and the level of pollution is below the threshold. In the second regime, both energy sources are used again but pollution is above the threshold. Only renewable resources are used in the third regime. The second part of this section focuses on corners regimes as specific cases that we compare to the general energy transition path to isolate the optimal path.

We exclude four corners regimes among a total of height corners regimes because they are unfeasible. The regimes of $T_1 = 0$ combined with $T_2 > 0$, $T_2 = 0$ or $T_2{=}\infty$ cannot occur because the economy cannot start above the pollution threshold without consuming the polluting energy. As the RE is a clean energy, if the economy starts with a RE, it will never cross the pollution threshold. Thus, the regime that corresponds to the case $T_2 = 0$ and $T_1 > 0$ is not possible. Finally, we work with the following corners regimes. The economy only switches on the energy regime ($T_1{=}\infty$), on the pollution regime ($T_2{=}\infty$), or the economy Never switches ($T_1{=}\infty$ and $T_2{=}\infty$). The economy can also start with the RE and never switches on the pollution regime ($T_1{=}\infty$ and $T_2{=}0$). In the last part of this section, we numerically solve for the optimal switching time associated with these corners regime and the general regime and provide the corresponding value function.

## 3.1 General energy transition path

In the present section, we analyze the general energy transition path that is described as the following. The economy starts using both sources of energy (RE and NRE) and therefore starts polluting. The economy accumulates the pollution up to the threshold $\overline{Z}$. Once the level of pollution exceeds this critical level $\overline{Z}$, the economy experiences a catastrophic event that could be a flooding for instance. The economy still uses both sources of energy before completely switching to the sole use of the clean energy. We backward solve for the optimal general path by starting from the third regime (sole use of RE) that is followed by the second regime and lastly by the first regime. We use the boundary conditions as in Boucekkine et al. (2013) to find the optimal time at which the economy will cross the critical pollution threshold and will turn to clean energy only. As it is not possible to get an analytical solution, we numerically solve it.

### 3.1.1 Third energy regime (RE, $\overline{Z}$)

During the third regime, the economy is no longer polluting because it stops using the dirty energy sources. The clean energy is the only available energy sources in the economy. Therefore, constraints on pollution accumulation and on NRE accumulation become both irrelevant. The economy has already crossed the critical pollution threshold and therefore still faces the negative consequences of the catastrophe. In this case, the social planner maximizes the sum of discounted utility subject to the constraint of capital accumulation.

The fact that the capital is split into final goods sector ($K_{2t}$) and clean energy sector ($K_{1t}$) together with the "Leontief conditions" in the final goods sector help deducing the following equation of capital accumulation (**see the proof in Appendix A**):

$$\dot{K} = \alpha_2(K - \frac{1}{\varphi}E_1 - \frac{1}{\varphi}E_2) - C$$

The social planner solves the following program:

$V_3 = Max \int_{T_2}^{\infty}(\frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta})e^{-\rho(t-T_2)}dt$
st $\dot{K} = \alpha_2(K - \frac{1}{\varphi}E_1 - \frac{1}{\varphi}E_2) - C$

The corresponding Hamiltonian is defined as:

$$H_3 = \frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta} + \lambda[\alpha_2(K - \frac{1}{\varphi}E_1 - \frac{1}{\varphi}E_2) - C],$$

with $\lambda$ the co-state variable related to capital $K$.

The first order conditions (FOCs) with respect to $C$, $E_2$ and $K$ respectively give:

$$C^{-\delta} = \lambda \tag{13}$$

$$E_2^{-\delta} = \frac{\alpha_2}{\varphi}\lambda \tag{14}$$

and

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha_2 \tag{15}$$

One can easily identify the consumption versus savings arbitrage condition in equations (13) and (14). It states that the marginal value of capital has to equal the marginal utility of consumption on the one hand and the marginal utility of energy services on the other hand. Moreover, condition (15) implies a constant instantaneous return over capital.

As energy services $E_1$ and capital $K_2$ are complementary in the final goods sector, we obtain:

$$Y = \alpha_2(K - \frac{1}{\varphi}E_1 - \frac{1}{\varphi}E_2) = \beta_2 E_1 \tag{16}$$

Using equations (13)-(16), we obtain:

$$\dot{K} = \frac{\alpha_2\beta_2\varphi}{\alpha_2 + \beta_2\varphi}K - [\frac{\alpha_2\beta_2}{\alpha_2 + \beta_2\varphi}(\frac{\alpha_2}{\varphi})^{-\frac{1}{\delta}} + 1]\lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2-\rho}{\delta})(t-T_2)}$$

The resolution of the above equation using the transversality condition **(see the proof in Appendix B)** gives:

$$K_t = -\frac{\Theta\delta}{\alpha_2 - \rho - \delta\Lambda}\lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2-\rho}{\delta})(t-T_2)},$$

where the unknown $\lambda_{T_2}$ will be determined in section 3.1.4 by using boundary conditions. We can easily deduce the value function $V_3$ during the third regime:

$$V_3 = -\frac{\delta[1+(\frac{\alpha_2}{\varphi})^{-\frac{1-\delta}{\delta}}]\lambda_{T_2}^{-\frac{1-\delta}{\delta}}}{(1-\delta)[\alpha_2(1-\delta)-\rho]}$$

Finally, we need to impose the non-negativity condition on $E_1$ so that:

$$E_1 = \frac{\alpha_2}{\alpha_2 + \varphi\beta_2}(\varphi K - E_2) > 0 \Leftrightarrow -\frac{\Theta\delta\varphi}{\alpha_2 - \rho - \delta\Lambda} - (\frac{\alpha_2}{\varphi})^{-\frac{1}{\delta}} > 0$$

### 3.1.2 Second energy regime (NRE-RE, $\overline{Z}$)

The economy uses both RE and NRE sources in the second energy regime. Despite the use of the dirty energy, the constraint on pollution accumulation is irrelevant because the economy has already crossed the critical pollution threshold. The economy starts facing damages from the catastrophe that occurred. The social planner maximizes the sum of the discounted post event utilities and the discounted value function of the third regime subject to constraints on capital accumulation and on NRE accumulation.

By using $Y = \alpha_2 K_2$ because of "Leontief conditions" and $K_2 = (1-\phi)K$, the equation of capital accumulation becomes:

$$\dot{K} = \alpha_2(1-\phi)K - C$$

The complementarity between the NRE and the RE, respectively gives that

$E_{1t} = \frac{1}{\xi}E_{1st} = E_{1xt}$ and $E_{2t} = \frac{1}{\xi}E_{2st} = E_{2xt}$. This implies that $E_{1st} = \xi E_{1t}$, and $E_{2st} = \xi E_{2t}$. By summing up the two precedent expressions, we get that $E_{st} = \xi(E_{1t} + E_{2t})$. The latter expression in (1) gives:

$$\dot{S}_t = -E_{st} = -\xi(E_{1t} + E_{2t})$$

The social planner solves the following program:

$$V_2 = Max \int_{T_1}^{T_2}(\frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta})e^{-\rho(t-T_1)} + V_3 * e^{-\rho T_2}$$

$$\text{st} \begin{cases} \dot{K} &= \alpha_2(1-\phi)K - C \\ \dot{S} &= -\xi(E_1 + E_2) \end{cases}$$

The corresponding Hamiltonian can be written as:

$$H_2 = \frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta} + \lambda_1[\alpha_2(1-\phi)K - C] - \lambda_2\xi(E_1 + E_2),$$

with $\lambda_1$ and $\lambda_2$ the co-state variable associated with the capital $K$ and the stock of the NRE $S_t$, respectively. The FOCs give:

$$C^{-\delta} = \lambda_1 \tag{17}$$

$$E_2^{-\delta} = \xi\lambda_2 \tag{18}$$

$$\frac{\dot{\lambda_1}}{\lambda_1} = \rho - \alpha_2(1-\phi) \tag{19}$$

and

$$\frac{\dot{\lambda_2}}{\lambda_2} = \rho \tag{20}$$

Like in the third regime, conditions (17) and (18) highlight the consumption (final goods and energy services) versus saving arbitrage conditions. Moreover, the instantaneous return over capital is constant as well as that of the NRE.

Using equations (17)-(20) we obtain **(see proof in Appendix C.1)**:

$$\dot{K} - \alpha_2(1-\phi)K = -\lambda_{1.T_1}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)} \tag{21}$$

The resolution of equation (21) gives:

$$K_t = -(\overline{K_2} - K_{T_1}) * e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)} + \overline{K_2},$$

where $\overline{K_2}$ is unknown and will be determined by using boundary conditions in section 3.1.4. Finally, by using the fact that the NRE is exhaustible and the fact that we have crossed the second pollution regime after a period of time $T_1$, we get **(see proof in Appendix C.2)**:

$$\lambda_{2.T_1} = \frac{1}{\xi}[(\frac{-S_0+\overline{Z}}{\xi} + \frac{\alpha_2(1-\phi)}{\beta_2}\overline{K_2}(T_2-T_1) - \frac{\alpha_2(1-\phi)\delta}{\beta_2(\alpha_2(1-\phi)-\rho)}(\overline{K_1}-K_0)e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1} * [e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}*(T_2-T_1)} - 1])\frac{\rho}{\delta[e^{-\frac{\rho}{\delta}T_1}-1]}]^{-\delta}$$

where $\lambda_{2.T_1}$, $K_{T_1}$ and $\overline{K_2}$ are unknown that will be determined by boundary conditions in section 3.1.4.

### 3.1.3 First energy regime (NR, $\underline{Z}$)

At the beginning of the program, the economy starts using both energy sources and faces a pollution accumulation constraint. A catastrophic event may happen as soon as the level of pollution reaches its critical threshold that results in a loss of capital during the following two regimes. We assume that the NRE is abundant ($S_0 > \overline{Z}$) so that we do not need to consider the accumulation of the NRE during the first regime. Therefore, the economy crosses the pollution threshold before the complete depletion of the NRE. The social planner maximizes the sum of the discounted pre-event utilities and the discounted value function of the second regime subject to the constraint on capital accumulation and that on the pollution accumulation. As we do not take into account of the regeneration capacity of the environment, the equation of the pollution accumulation can be expressed as the opposite of that of the NRE: $\dot{Z} = -\dot{S} = \xi(E_1+E_2)$ .

The social planner solves:

$V_1 = Max \int_0^{T_1}[(\frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta})e^{-\rho t}]dt + V_2^* e^{-\rho T_1}$

$$\text{st} \begin{cases} \dot{K} &= \alpha_2(1-\phi)K - C \\ \dot{Z} &= \xi(E_1+E_2) \end{cases}$$

The Hamiltonian can be written as:

$$H_1 = \frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta} + \lambda_1[\alpha_2(1-\phi)K - C] + \lambda_2\xi(E_1+E_2),$$

with $\lambda_1$ and $\lambda_2$ the co-state variables associated to the capital $K$ and to the stock of pollution $Z_t$, respectively. The FOCs give:

$$C^{-\delta} = \lambda_1 \tag{22}$$

$$E_2^{-\delta} = -\xi\lambda_2 \tag{23}$$

$$\frac{\dot{\lambda}_1}{\lambda_1} = \rho - \alpha_2(1-\phi) \tag{24}$$

$$\frac{\dot{\lambda}_2}{\lambda_2} = \rho \tag{25}$$

FOCs (22)-(25) give**(see proof in Appendix C.3)**:

$$\dot{K} - \alpha_2(1-\phi)K = -\lambda_{1.0}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t}$$

We solve the above equation to get:

$$K_t = -(\overline{K_1} - K_0)e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t} + \overline{K_1}$$

Finally, we use the fact that at the end of the first regime, we cross the pollution threshold and we obtain:

$\frac{1}{\xi}\overline{Z} = -(-\lambda_{2.0}\xi)^{-\frac{1}{\delta}} * \frac{\delta}{\rho}[e^{-\frac{\rho}{\delta}T_1} - 1] + \frac{\alpha_2(1-\phi)}{\beta_2}\overline{K_1} * T_1 - \frac{\alpha_2(1-\phi)\delta}{\beta_2(\alpha_2(1-\phi)-\rho)}(\overline{K_1} - K_0)\left[e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta}) * T_1} - 1\right]$

This implies that:

$\lambda_{2.0} = -\frac{1}{\xi}[(-\frac{\overline{Z}}{\xi} + \frac{\alpha_2(1-\phi)}{\beta_2}\overline{K_1} * T_1 - \frac{\alpha_2(1-\phi)\delta}{\beta_2(\alpha_2(1-\phi)-\rho)}(\overline{K_1} - K_0)[e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta}) * T_1} - 1]) * \frac{\rho}{\delta[e^{-\frac{\rho}{\delta}T_1} - 1]}]^{-\delta}$

where $\lambda_{2.0}$ and $\overline{K_1}$ are unknown and will be determined in section 3.1.4 using boundary conditions.

### 3.1.4 Boundary conditions

Following Boucekkine et alii (2013), we use three types of boundary conditions: continuity of $\lambda_1$ , continuity of $K$ and the equality of the Hamiltonian at the switching time. The co-state variable $\lambda_2$ associated with the pollution stock $Z$ is not continuous at the switching time $T_1$ because Z is fixed to $\overline{Z}$. At the switching time $T_2$, $Z$ can be freely chosen and becomes continuous but it no longer exist during the third regime because the RE is not polluting. The continuity of $\lambda_1$ together with that of $K$ help determine $\overline{K}_1$, $\overline{K}_2$, $K_{T_1}$, $K_{T_2}$, $\lambda_{T_2}$, $\lambda_{2.0}$ and $\lambda_{2.T_1}$ (**see the proof in Appendix E**). We then simultaneously and numerically solve the equality of Hamiltonians at the switching time $T_1$ and $T_2$ to get $T_1$ and $T_2$. Now, let us look at corners regimes before providing the numerical value function.

## 3.2 Corners regimes

As we exclude four corners regimes among a total of height corners regimes because they are unfeasible, we present in this section only the four relevant ones.

### 3.2.1 Sole switch to the energy regime ($T_1$=$\infty$)

This case is a corner regime in which the economy never exceeds the critical pollution threshold and therefore only switches to the sole adoption of the clean energy. The economy starts using both oil and the renewable resource that are complementary and the pollution is below its critical level. After some time $T$, it switches to the sole use of renewable energy before the level of pollution crosses the pollution threshold. Therefore, the economy escapes the catastrophe forever. To get the switching time $T$, it is sufficient to set $T_1$=$\infty$ and $T_2 = T$.

### 3.2.2 Sole switch above the pollution threshold level ($T_2$=$\infty$)

This case corresponds to the transition from the first regime to the second regime without the switch to the third regime. Again, the economy starts using both oil and the renewable source of energy with a level of pollution that is below the threshold level. Then, the economy switches to the regime in which both energy sources are still used but now, the level of pollution is above its critical threshold and the economy never adopts the renewable energy. To get the switching time $T$ and the dynamics of variables, one just needs to set $T_2$=$\infty$ and $T_1 = T$.

### 3.2.3 No switch ($T_1$=$\infty$ and $T_2$=$\infty$)

On the no switch transition path, the economy always uses both oil and the renewable source of energy. Moreover, it never solely uses the RE and the level of pollution remains below its critical threshold forever. This energy transition path corresponds to the first regime and one does not need to use boundary conditions to get the switching time. It is sufficient to set $T_1$=$\infty$ and use the transversality conditions that give $\overline{K}_1 = 0$.

### 3.2.4 Starting with RE ($T_1$=$\infty$ and $T_2$=0)

On this transition path, the economy never uses the NRE and therefore does not pollute. The pollution threshold becomes then irrelevant. It corresponds to the third regime without any pollution threat. In this case, we just need to set $T_1$=$\infty$ and $T_2$=0.

# 4 Numerical results

In this section, we numerically solve for the switching times $T_1$ and $T_2$ and calculate the value functions of the general energy path and that of each corner regimes. We present the parameter values that are used to get the numerical results. We also provide the numerical value functions and the sensitivity analysis.

## 4.1 Parameter values

The parameters values are chosen as follows. As we are concerned about environmental issues (pollution) that can lead to catastrophic event, we set the discount rate $\rho$ to 0.05, so that people are more patient and concerned about the consequences of their behaviour in the future. This value of 0.05 is standard in economics literature. The sensitivity analysis will help to point out how the pollution would evolve in the case people are impatient (high discount factor). We consider an initial stock of the NRE $S_0$ equals to 28000 and the pollution threshold $\overline{Z} = 1000$ as a benchmark. In the final goods sector, we set the parameter $\alpha_2$ that is related to capital to 0.0001 and that of energy $\beta_2$ to 0.02 in the Leontief function. The factor of capital transformation into energy $\varphi$ is set to 1.5. We also assume that to get one unit of energy services, the economy should provide 1.5 units of the NRE such that $\xi$=1.5. The part of capital that is lost due to the catastrophe $\theta$ is set to 0.05. The value of the degree of relative risk aversion $\delta$ and that of the initial level of capital are set to 2 and 95, respectively. Finally, we take $\phi$=0.1.

We compare the value functions among them and find out the optimal one that gives the highest value function. We also perform a sensitivity analysis to check for the impact of each parameters on the switching time.

## 4.2 Value functions

The numerical results are summarized in the table below.

| **Energy transition path** | **values functions** |
|---|---|
| General energy transition path | –36.7722 |
| $T_1$=$\infty$, | -42.4142 |
| $T_2$=$\infty$ | **-17.6568** |
| $T_1$=$\infty$ *and* $T_2$=$\infty$ | *-19.2455* |
| $T_1$=$\infty$ and $T_2$=0 | -50.8961 |

**Table 1**: Numerical results

The optimal energy transition path is the one that gives the highest value function to the planner. It corresponds to $T_2=\infty$ corner regime. It means that the optimal energy transition path can be described as follows. The economy starts using both sources of energy. Then, it crosses the pollution threshold and loses a part of its capital. The sole adoption of the RE is never optimal for the economy. One could notice that the corner regime that corresponds to $T_1=\infty$ and $T_2=\infty$ case is close to that of the optimal one. This may be justified by the fact that the economy does not lose/gain enough by refraining to pollute more in order to never cross the pollution threshold. One could also remark that the general energy transition path is far from being the optimal one. Hence, we can conclude that it exists numerical values of the parameters that imply that it is not be optimal to only adopt RE. This surprising result goes in line with the asymptotic energy transition arguments that state that the complete transition to the sole use of clean energy may happen only in the long run.

### 4.3 Sensitivity analysis

We summarize the sensitivity analysis in the table below.

| | $\rho$ | $\theta$ | $S_0$ | $\alpha_2$ | $\beta_2$ | $\overline{Z}$ |
|---|---|---|---|---|---|---|
| $T_1$ | - | + | small effect | + | + | + |

**Table 2**: sensitivity analysis

Sensitivity analysis shows that the optimal time to cross the pollution threshold positively depends on (i) the corresponding capital loss, (ii) the productivity of capital and energy services, (iii) the level of the pollution threshold. Moreover, it negatively depends on the discount rate.

More impatient social planner is willing to extract more NRE and will quickly cross the critical pollution threshold. As the damage from the catastrophic event is high, people will fear about the negative consequences of their "dirty energy" use and will reduce it. This could help them staying longer in the first regime before crossing the pollution threshold. Likewise, people will reduce their use of energy resource as long as the productivity of capital and energy services is high. The higher is the pollution threshold that could provoke the catastrophic event, the more people will stay in the pre-event regime. Finally, the initial stock of the NRE does not matter. The social planner mainly cares about the management of the initial stock of the NRE so that the switching time $T_1$ is robust with respect to the initial stock of the NRE.

## 5 Introducing Investment in Energy Saving Technologies (EST)

Let us recall that $E_1$ and $E_2$ are energy services in the final goods sector and for households respectively. The final goods sector uses $E_{1s}$ of the NRE and $E_{1x}$ of the RE, while households use $E_{2s}$ of the NRE and $E_{2x}$ of the RE. At each period of time, in addition to consumption and investments in energy sector and final goods sector, the economy now invests a part of the final goods production $q_t$ in energy savings technologies. We assume that the investment $q_t$ serves to reduce by $\vartheta(q)$ units the resources that the economy needs in order to get the same energy services. Implicitly, it means that we do not account for a scale effect[2]. Let us assume that $\vartheta(q)$ is an increasing function ($\vartheta^{'}(q) > 0$) and exhibits decreasing marginal returns ( $\vartheta^{''}(q) < 0$) in the abatement investment. $\vartheta(q)$ is increasing in the sense that the more the economy invests in EST, the more it reduces the use of the energy resource to get a given energy services. Moreover, as $var\theta(q)$ is increasing, we assume that $\vartheta(q)$ is concave in order to have a maximum for $\vartheta(q)$. Also, we avoid a complete elimination of the use of energy resources so that it will require an infinite amount of investment to do so ( $\underset{q\to\infty}{lim} E_i - \vartheta(q) = 0$ with i $\in \{1, 2\}$ and $\vartheta(0) = 0$ ).

Due to the investment $q_t$ in energy saving technologies, the dynamics of capital, of the NRE and that of pollution are modified, while the utility of the household remains the same (**see the proof in Appendix D**). Note that those dynamics do not change in terms of the extraction of energy resource, but only in terms of the energy services. The same amount of energy resource provides more energy services when the energy saving technologies are used. In comparison with the previous model, the social planner has to consider one additional control variable (investment $q_t$) to solve for the optimal energy transition. The main changes in the results are the following (**see the proof in Appendix F**). The level of capital at each period of time during the three regimes has an additional negative component. In fact, the economy additionally uses a part of income to invest in EST. This part could have been invested in productive sector (final goods and energy) or consumed by households. As the share of the income that goes to investment is reduced by investment in energy saving technologies, one should expect a decrease in capital.

As before, we also discuss corner regimes that we compare to the general energy transition path to isolate the optimal one. In order to make our numerical results comparable, we use the same set parameters values as before. Additionally, we set the productivities of investment in EST both at household level and at industry level $\sigma$ to 0.2. We numerically solve for the switching times $T_1$ and $T_2$ and calculate the value functions of the general energy path and that of each corner regimes. We compare the value functions among them and find out the optimal one that gives the highest value function. We also perform a sensitivity analysis to check for the impact of each parameters on the switching times. The numerical results are threefold. First, investments in energy saving technologies do not alter the optimal energy transition path which remains the asymptotic transition to the sole use of renewable energy. The second point is that investments in energy saving technologies increase the time at which the economy may experience the catastrophe and that of the sole adoption of the renewable energy. In fact, investments in energy saving technologies help to reduce the consumption of energy for the same quality of energy services and therefore help to reduce the pollution. Although

[2]One should also consider that the investment $q_t$ induces a scale effect. This makes the present model very complex and unsolvable because of the interaction that may appear between $q_t$ and all the precedent control variables like the energy services.

investment in EST reduces the share of the income that goes to investment in both the final goods sector and the renewable energy production sector, it favours the energy transition. Last but not least, investment in EST increases the welfare of the society. Then, it is profitable for the economy to combine both adoption of renewable energy and investments in energy savings technologies.

# 6 Conclusion

This paper makes four contributions. First, it proposes a general appraisal of optimal switching problems of energy transition exhibiting (i) the possibility of a catastrophe due to pollution accumulation, and (ii) technological regimes with the adoption of renewable energy. Secondly, it applies the optimal control material in Boucekkine et al. (2013) to address the problem of the optimal energy transition. We solve the model backward by using the analytical first order optimality conditions. This paper numerically show that the optimal energy transition path may correspond to a corner regime in which the economy starts using both resources, crosses the pollution threshold by losing a part of its capital and never adopts a regime with only renewable energy. This result goes in line with the asymptotic energy transition argument stating that the transition to a clean energy may happen only in the long run. The fourth contribution of this paper is to extend this model to the adoption of energy savings technologies. We mainly find that investments in energy saving technologies do not alter the optimal energy transition path. Moreover, investments in energy saving technologies favour the energy transition in the sense that it increases the time at which the economy may experience the catastrophe and that of the sole adoption of the renewable energy. Finally, as investment in EST increases the welfare of the society, it is then profitable to combine both adoption of renewable energy and investments in energy savings technologies. However, investments in EST reduce the share of the income that goes to investment in both the final goods sector and the renewable energy production sector.

As policy implications, we could recommend that it is important to adopt some economic instruments such as taxes on the "dirty" energy, subsidies on the "clean" energy or incentives for energy saving technologies in order to promote the energy transition. Also, it is profitable to design economic instruments that jointly target the promotion of clean energy and incentives for investments in energy saving technologies. But those economic instruments should be designed to meet the requirements of a transition to a sole use of clean energy in the long run. In fact, as a quick and full transition is not optimal for the economy, we should not expect any immediate transition to an economy that only uses the renewable sources of energy. The present paper can be extended to investigate the optimal taxes/subsidies that may favour the adoption of renewable energy and the investments in energy savings technologies in line with the asymptotic energy transition result.

# 7 References

[1] Acemoglu, D., Akcigit, U., Hanley, D., Kerr, W., 2014. The transition to clean technology. MIT Department of Economics, Working Paper.

[2] Amigues J.P., Ayong Le Kama A., Chakravorty U., Moreaux M., 2013. Equilibrium Transitions from Non Renewable Energy to Renewable Energy under Capacity Constraints. TSE Working Paper, 13-440.

[3] Ayong Le Kama, A., Pommeret, A., Prieur, F., 2011. Optimal emission policy under the risk of irreversible pollution. Nota di lavoro/Fondazione Eni Enrico Mattei:Sustainable development.

[4] Berndt, E. R., Wood, D. O., 1975. Technology, prices, and the derived demand for energy. The review of Economics and Statistics, 259-268.

[5] Boucekkine, R., Pommeret, A., 2004. Energy saving technical progress and optimal capital stock: the role of embodiment. Economic Modelling 21(3), 429-444.

[6] Boucekkine, R., Pommeret, A., Prieur, F., 2013. Optimal regime switching and threshold effects. Journal of Economic Dynamics and Control 37(12), 2979-2997.

[7] Chakravorty, U., Leach, A., Moreaux, M., 2012. Cycles in non renewable resource prices with pollution and learning-by-doing. Journal of Economic Dynamics and Control 36 (10), 1448–1461.

[8] Chakravorty, U., Magné, B., Moreaux, M., 2006. A Hotelling model with a ceiling on the stock of pollution. Journal of Economic Dynamics and Control 30(12), 2875–2904.

[9] Chakravorty, U., Moreaux, M., Tidball, M., 2008. Ordering the extraction of polluting non renewable resources. The American Economic Review 98(3), 1128–1144.

[10] Chakravorty, U., Roumasset, J., Tse, K., 1997. Endogenous substitution among energy resources and global warming. Journal of Political Economy, 105(6), 1201-1234.

[11] Charlier, D., Mosino, A., Pommeret, A., 2011. Energy-saving technology adoption under uncertainty in the residential sector. Annals of Economics and Statistics/Annales d'Économie et de Statistique, 43-70.

[12] Clarke, H. R., Reed, W. J., 1994. Consumption/pollution tradeoffs in an environment vulnerable to pollution-related catastrophic collapse. Journal of Economic Dynamics and Control, 18(5), 991-1010.

[13] Cropper, M. L., 1976. Regulating activities with catastrophic environmental effects. Journal of environmental Economics and Management, 3(1), 1-15.

[14] Dasgupta, P., Heal, G., 1979. Economic theory and exhaustible resources. Cambridge University Press.

[15] Dasgupta, P., Heal, G., 1974. The optimal depletion of exhaustible resource. The Review of Economic Studies 41, 3-28.

[16] Dasgupta, P., Stiglitz, J., 1981. Resource depletion under technological uncertainty. Econometrica: Journal of the Econometric Society, 85-104.

[17] De Groot, H. L., Verhoef, E. T., Nijkamp, P., 2001. Energy saving by firms: decision-making, barriers and policies. Energy Economics 23(6), 717-740.

[18] Díaz, A., Puch, L. A., 2013. A Theory of Vintage Capital Investment and Energy Use. Working paper, Documentos de Trabajo del Instituto Complutense de Análisis Económico (ICAE) 35,23330.

[19] Forster, B., A., 1975. Optimal pollution control with a nonconstant exponential rate of decay. Journal of environmental Economics and Management, 2(1), 1-6.

[20] GEA, 2012. Global Energy Assessment- Toward a Sustainable Future. Cambridge University Press, Cambridge UK and New York, NY, USA and the International Institute for Applied Systems Analysis, Laxenburg, Austria.

[21] Gjerde, J., Grepperud, S., Kverndokk, S., 1999. Optimal climate policy under the possibility of a catastrophe. Resource and Energy Economics, 21(3), 289-317.

[22] Golosov, M., Hassler, J., Krusell, P., Tsyvinski, A., 2011. Optimal taxes on fossil fuel in general equilibrium. Working paper, National Bureau of Economic Research.

[23] Horii, R., Ikefuji, M., 2010. Natural disasters in a two-sector model of endogenous growth. Center Discussion Paper, Economic Growth Center.

[24] Karp, L., Tsur, Y., 2011. Time perspective and climate change policy. Journal of Environmental Economics and Management, 62(1), 1-14.

[25] Kollenbach, G., 2013. Endogenous Growth with a Ceiling on the Stock of Pollution. Working paper, MPRA.

[26] Krautkraemer, J., A., 1998. Nonrenewable resource scarcity. Journal of Economic Literature, 2065-2107.

[27] Lafforgue, G., Magné, B., Moreaux, M., 2008. Optimal Sequestration Policy with a Ceiling on the Stock of Carbon in the Atmosphere. Working paper No. 401. Institut d'Économie Industrielle (IDEI), Toulouse.

[28] Nævdal, E., 2006. Dynamic optimization in the presence of threshold effects when the location of the threshold is uncertain – with an application to a possible disintegration of the western antarctic ice sheet. Journal of Economic Dynamics and Control 30, 1131–1158.

[29] Nordhaus, W., D., 1994. Managing the global commons: the economics of climate change. Cambridge, MIT press, 101-120.

[30] Pelli M., 2012. The Elasticity of Substitution between Clean and Dirty Inputs in the Production of Electricity. Mimeo, University of Alberta.

[31] Pindyck, R. S., 2002. Optimal timing problems in environmental economics. Journal of Economic Dynamics and Control 26.9, 1677-1697.

[32] Pindyck, R. S., Rotemberg, J. J., 1983. Dynamic factor demands and the effects of energy price shocks. The American Economic Review, 1066-1079.

[33] Pommeret, A., Prieur, F., 2009. Double irreversibility and environmental policy design. Working paper, No. 2009.10.

[34] Prieur, F., 2007. The Environmental Kuznets Curve in a World of Irreversibility. Working paper, Nota di Lavoro, Fondazione Eni Enrico Mattei, No. 57.2007.

[35] Prieur, F., Tidball, M., Withagen, C., 2013. Optimal emission-extraction policy in a world of scarcity and irreversibility. Resource and Energy Economics 35(4), 637-658.

[36] Tahvonen, O., 1996. Trade with Polluting Non renewable Resources. Journal of Environmental Economics and Management 30, 1-17.

[37] Tahvonen, O., 1997. Fossil Fuels, Stock Externalities, and Backstop Technology. The Canadian Journal of Economics 30, 855-874.

[38] Tahvonen, O., Withagen, C., 1996. Optimality of irreversible pollution accumulation. Journal of Economic Dynamics and Control 20(9), 1775-1795.

[39] Tsur, Y., Withagen, C., 2013. Preparing for catastrophic climate change. Journal of Economics 110(3), 225-239.

[40] Tsur, Y., Zemel, A., 1996. Accounting for global warming risks: resource management under event uncertainty. Journal of Economic Dynamics and Control, 20(6), 1289-1305.

[41] Tsur, Y., Zemel, A., 2003. Optimal transition to backstop substitutes for nonrenewable resources. Journal of Economic Dynamics and Control, 27.4, 551-572.

[42] Tsur, Y., Zemel, A., 2005. Scarcity, growth and R&D. Journal of Environmental Economics and Management 49(3), 484-499.

[43] Ulph, A., Ulph, D., 1997. Global warming , irreversibility and learning. The Economic Journal, 107(442), 636-650.

[44] Van der Ploeg, F., Withagen, C., 2012. Is there really a Green Paradox?. Journal of Environmental Economics and Management 64(3), 342-363.

[45] Van der Ploeg, F., Withagen, C., 2014. Growth, renewables, and the optimal carbon tax. International Economic Review, 55.1, 283-311.

# 8 Appendix

## 8.1 Appendix A

Let us recall that the equation of capital accumulation is:

$$\dot{K}_t = Y_t - C_t \tag{26}$$

We also know that:

$$Y = min\{\alpha_2 K_2, \beta_2 E_{1t}\},$$

Where,

$$K = K_1 + K_2$$

and

$$E_x = \varphi K_1$$

$E_x = \varphi K_1$ implies that $K_1 = \frac{E_x}{\varphi}$.
Then,

$$K_2 = K - K_1 = K - \frac{E_x}{\varphi} \tag{27}$$

From "Leontief conditions" in final goods sector, we have that:

$$Y = \alpha_2 K_2 = \beta_2 E_{1t} \tag{28}$$

During the third regime, only the RE is used so that we have the following equalities:

$$E_{1xt} = E_{1t},$$

and

$$E_{2xt} = E_{2t}.$$

By summing up the two above expressions, we get that:

$$E_{xt} = E_{1t} + E_{2t}. \tag{29}$$

Plugging (38) into (35) gives:

$$K_2 = K - \frac{(E_{1t} + E_{2t})}{\varphi} \tag{30}$$

Let put (39) into (36) and get:

$$Y = \alpha_2 K - \alpha_2 \frac{E_{1t} + E_{2t}}{\varphi} \tag{31}$$

Finally, use (40) in the equation of capital accumulation (26) to get that:

$$\dot{K}_t = \alpha_2 K_t - \alpha_2 \frac{E_{1t} + E_{2t}}{\varphi} - C_t.$$

## 8.2 Appendix B

To determine the expression of capital in the third regime, we need to solve the following equation of capital accumulation for the capital $K_t$:

$$\dot{K} = \Lambda K - \Theta \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2 - \rho}{\delta})(t - T_2)}$$

where $\Lambda = \frac{\alpha_2 \beta_2 \varphi}{\alpha_2 + \beta_2 \varphi}$ and $\Theta = \frac{\alpha_2 \beta_2}{\alpha_2 + \beta_2 \varphi}(\frac{\alpha_2}{\varphi})^{-\frac{1}{\delta}} + 1$.

First, let us make a change of variables as follows:

$$x = K e^{-\Lambda(t - T_2)}.$$

We have that:

$$\dot{x} e^{\Lambda(t - T_2)} = \dot{K} - \Lambda K = -\Theta \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2 - \rho}{\delta})(t - T_2)}$$

This implies that

$$\dot{x} = -\Theta \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2 - \rho}{\delta} - \Lambda)(t - T_2)}$$

The solution of the above equation is:

$$x_t = -\frac{\Theta \delta}{\alpha_2 - \rho - \delta \Lambda} \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2 - \rho}{\delta} - \Lambda)(t - T_2)} + \overline{x},$$

Hence,

$$K_t = -\frac{\Theta \delta}{\alpha_2 - \rho - \delta \Lambda} \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2 - \rho}{\delta})(t - T_2)} + \overline{K}_3,$$

where $\overline{K}_3 = \overline{x}e^{\Lambda(t-T_2)}$.

We now use the transversality conditions to determine $\overline{K}_3$:

$$\lim_{t\to\infty} \lambda_t K_t e^{-\rho(t-T_2)} = 0$$

This implies that $\lim_{t\to\infty} \overline{x}\lambda_{T_2} = 0$, for $\alpha_2(1-\delta) < \rho$ with $\overline{K}_3 = \overline{x}e^{\alpha_2(t-T_2)}$. This is true if and only if $\overline{x} = 0$ and therefore $\overline{K}_3 = 0$. $\overline{K}_3 = 0$ implies that:

$$K_t = -\frac{\Theta\delta}{\alpha_2 - \rho - \delta\Lambda} \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2-\rho}{\delta})(t-T_2)}.$$

## 8.3 Appendix C.1

The expression of capital in the second regime is determined from FOCs as follows.

From (19) and (20), we deduce that:

$$\lambda_1 = \lambda_{1.T_1} e^{(\rho-\alpha_2(1-\phi))(t-T_1)} \tag{32}$$

and

$$\lambda_2 = \lambda_{2.T_1} e^{\rho(t-T_1)} \tag{33}$$

(32) in (17) and (33) in (18) respectively lead to:

$$C_t = \lambda_{1.T_1}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)}$$

and

$$E_{2t} = (\xi\lambda_{2.T_1})^{-\frac{1}{\delta}} e^{-\frac{\rho}{\delta}(t-T_1)}$$

By using the above expression of C, the equation of capital accumulation becomes:

$$\dot{K} - \alpha_2(1-\phi)K = -\lambda_{1.T_1}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)}$$

By using the same variable change as in Appendix B, the solution of the above equation is:

$$K_t - \overline{K_2} = -\lambda_{1.T_1}^{-\frac{1}{\delta}} * \frac{\delta}{\alpha_2(1-\phi)(1-\delta)-\rho} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)}.$$

To determine $\lambda_{1.T_1}$, let us take $K_t$ at t=$T_1$:

$$K_{T_1} - \overline{K_2} = -\lambda_{1.T_1}^{-\frac{1}{\delta}} * \frac{\delta}{\alpha_2(1-\phi)(1-\delta)-\rho}.$$

This implies that:

$$\lambda_{1.T_1}^{-\frac{1}{\delta}} = (\overline{K_2} - K_{T_1}) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}$$

Plugging this last expression into $K_t$ gives:

$$K_t = -(\overline{K_2} - K_{T_1}) * e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)} + \overline{K_2},$$

where $\overline{K_2}$ is unknown and will be determined by using boundary conditions in section 3.1.4.

## 8.4 Appendix C.2

We assume that the NRE is exhaustible and we have crossed the second regime after a period of time $T_1$. Then, the initial stock of the NRE $S_0$ is equal to the sum of the part of the NRE that is used during the first regime which corresponds to the total amount of pollution $\overline{Z}$ and the part of the NRE that the economy uses during the second regime. We have that:

$$S_0 = \underbrace{\int_0^{T_1} \xi(E_{1t} + E_{2t})dt}_{\overline{Z}} + \int_{T_1}^{T_2} \xi(E_{1t} + E_{2t})dt$$

This implies that:

$$S_0 - \overline{Z} = \int_{T_1}^{T_2} \xi(E_{1t} + E_{2t})dt = \frac{\xi\alpha_2}{\beta_2}\int_{T_1}^{T_2} K_t dt + \xi\int_{T_1}^{T_2} E_{2t}dt$$

with $S_0 > \overline{Z}$.

Let solve separately each part of the the Right Hand Side (RHS) of the above equation:

$$\chi_1 = \frac{\xi\alpha_2}{\beta_2}\int_{T_1}^{T_2} K_t dt$$

and

$$\chi_2 = \xi\int_{T_1}^{T_2} E_{2t}dt.$$

We get that:

$$\chi_1 = \frac{\xi\alpha_2}{\beta_2}\int_{T_1}^{T_2}[-(\overline{K_2} - K_{T_1}) * e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)}]dt + \frac{\xi\alpha_2}{\beta_2}\int_{T_1}^{T_2}\overline{K_2}dt,$$

$$\Rightarrow \chi_1 = -\frac{\xi\delta\alpha_2}{\beta_2(\alpha_2(1-\phi)-\rho)} * (\overline{K_2} - K_{T_1}) * \left[e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(T_2-T_1)} - 1\right] + \frac{\xi\alpha_2(1-\phi)}{\beta_2}\overline{K_2}\left[(T_2 - T_1)\right]$$

and

$$\chi_2 = \xi\int_{T_1}^{T_2}(\xi\lambda_{2.T_1})^{-\frac{1}{\delta}}e^{-\frac{\rho}{\delta}(t-T_1)}dt$$

$$\Rightarrow \chi_2 = -\xi(\xi\lambda_{2.T_1})^{-\frac{1}{\delta}} * \frac{\delta}{\rho}\left[e^{-\frac{\rho}{\delta}(T_2-T_1)} - 1\right]$$

Using the above expressions in $S_0 - \overline{Z} = \chi_1 + \chi_2$, we get that:

$\frac{1}{\xi}(S_0 - \overline{Z}) = -(\xi\lambda_{2.T_1})^{-\frac{1}{\delta}} * \frac{\delta}{\rho}\left[e^{-\frac{\rho}{\delta}(T_2-T_1)} - 1\right] + \frac{\alpha_2(1-\phi)}{\beta_2}\overline{K_2}\left[(T_2 - T_1)\right] - \frac{\delta\alpha_2}{\beta_2(\alpha_2(1-\phi)-\rho)} * (\overline{K_2} - K_{T_1}) * \left[e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(T_2-T_1)} - 1\right]$

## 8.5 Appendix C.3

The level of capital at each time during the first regime is determined as follows. (**??**) in (22) and (**??**) in (23) respectively lead to:

$$C_t = \lambda_{1.0}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t}$$

and

$$E_{2t} = (-\lambda_{2.0}\xi)^{-\frac{1}{\delta}} e^{-\frac{\rho}{\delta}t}$$

As before we also replace the expression of $C$ into the equation of capital accumulation to get:

$$\dot{K} - \alpha_2(1-\phi)K = -\lambda_{1.0}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t}$$

We solve the above equation to get:

$$K_t - \overline{K_1} = -\lambda_{1.0}^{-\frac{1}{\delta}} * \frac{\delta}{\alpha_2(1-\phi)(1-\delta)-\rho} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t}$$

At t=0, $K_0 = -\lambda_{1.0}^{-\frac{1}{\delta}} * \frac{\delta}{\alpha_2(1-\phi)(1-\delta)-\rho} + \overline{K_1}$. This implies that:

$$\lambda_{1.0}^{-\frac{1}{\delta}} = (\overline{K_1} - K_0) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}.$$

Replacing this expression into the expression of $K_t$, we get:

$$K_t = -(\overline{K_1} - K_0)e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t} + \overline{K_1}$$

At the end of the first regime, we cross the pollution threshold so that:

$$\overline{Z} = \int_0^{T_1} \xi(E_{1t} + E_{2t})dt$$

By replacing $E_{1t}$ and $E_{2t}$ as in the second regime, we get:

$$\frac{1}{\xi}\overline{Z} = -(-\lambda_{2.0}\xi)^{-\frac{1}{\delta}} * \frac{\delta}{\rho}[e^{-\frac{\rho}{\delta}T_1} - 1] + \frac{\alpha_2(1-\phi)}{\beta_2}\overline{K_1} * T_1 - \frac{\alpha_2(1-\phi)\delta}{\beta_2(\alpha_2(1-\phi)-\rho)}(\overline{K_1} - K_0)\left[e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})*T_1} - 1\right]$$

This implies that:

$$\lambda_{2.0} = -\frac{1}{\xi}[(-\frac{\overline{Z}}{\xi} + \frac{\alpha_2(1-\phi)}{\beta_2}\overline{K_1} * T_1 - \frac{\alpha_2(1-\phi)\delta}{\beta_2(\alpha_2(1-\phi)-\rho)}(\overline{K_1} - K_0)[e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})*T_1} - 1]) * \frac{\rho}{\delta[e^{-\frac{\rho}{\delta}T_1} - 1]}]^{-\delta}$$

where $\lambda_{2.0}$ and $\overline{K_1}$ are unknown and will be determined in section 3.1.4 using boundary conditions.

## 8.6 Appendix D

The equation of capital accumulation becomes:

$$\dot{K}_t = Y_t - C_t - q_t \tag{34}$$

We also know that:

$$Y = min\{\alpha_2 K_2, \beta_2 E_{1t}\},$$

Where,

$$\begin{cases} E_{1t} = & min\{\frac{1}{\xi} E_{1st}, E_{1xt}\} + \theta_1(q_t), & t < T_2 \\ E_{1t} = & E_{1xt} + \theta_1(q_t), & t > T_2 \end{cases}$$

$t < T_2$ corresponds to the first two regimes in which both types of energy are simultaneously used, while $t > T_2$ denotes the third regime that uses only the RE.

$$K = K_1 + K_2$$

and

$$E_x = \varphi K_1$$

Likewise, $E_{2t}$ is defined as:

$$\begin{cases} E_{2t} = & min\{\frac{1}{\xi} E_{2st}, E_{2xt}\} + \theta_2(q_t), & t < T_2 \\ E_{2t} = & E_{2xt} + \theta_2(q_t), & t > T_2 \end{cases}$$

$E_x = \varphi K_1$ implies that $K_1 = \frac{E_x}{\varphi}$.
Then,

$$K_2 = K - K_1 = K - \frac{E_x}{\varphi} \tag{35}$$

From "Leontief conditions" in final goods sector, we have that:

$$Y = \alpha_2 K_2 = \beta_2 E_{1t} \tag{36}$$

Using the complementarity between the NRE and RE, we respectively get that:

$$\begin{cases} E_{1t} - \theta_1(q_t) = & \frac{1}{\xi} E_{1st} = E_{1xt}, & t < T_2 \\ E_{1t} - \theta_1(q_t) = & E_{1xt}, & t > T_2 \end{cases}$$

and

$$\begin{cases} E_{2t} - \theta_2(q_t) = & \frac{1}{\xi} E_{2st} = E_{2xt}, & t < T_2 \\ E_{2t} - \theta_2(q_t) = & E_{2xt}, & t > T_2 \end{cases}$$

This implies that:

$$\begin{cases} E_{1st} = & \xi(E_{1t} - \theta_1(q_t)), & t < T_2 \\ E_{1st} = & 0, & t > T_2 \end{cases},$$

$$\begin{cases} E_{2st} = & \xi(E_{2t} - \theta_2(q_t)), & t < T_2 \\ E_{2st} = & 0, & t > T_2 \end{cases},$$

$$E_{1xt} = E_{1t} - \theta_1(q_t), \nabla\ t,$$

and

$$E_{2xt} = E_{2t} - \theta_2(q_t), \nabla\ t.$$

By summing up the four above expressions, we get that:

$$\begin{cases} E_{st} = & \xi(E_{1t} + E_{2t}) - \xi(\theta_1(q_t) + \theta_2(q_t)), & t < T_2 \\ E_{st} = & 0, & t > T_2 \end{cases} \tag{37}$$

and

$$E_{xt} = (E_{1t} + E_{2t}) - (\theta_1(q_t) + \theta_2(q_t)), \nabla\ t. \tag{38}$$

Plugging (38) into (35) gives:

$$K_2 = K - \frac{(E_{1t} + E_{2t}) - (\theta_1(q_t) + \theta_2(q_t))}{\varphi} \tag{39}$$

Let put (39) into (36) and get:

$$Y = \alpha_2 K - \alpha_2 \frac{(E_{1t} + E_{2t}) - (\theta_1(q_t) + \theta_2(q_t))}{\varphi} \tag{40}$$

Finally, to get the equation of capital accumulation in the third regime let us use (40) in the equation of capital accumulation (34) to get that:

$$\dot{K}_t = \alpha_2 K_t - \alpha_2 \frac{(E_{1t} + E_{2t}) - (\theta_1(q_t) + \theta_2(q_t))}{\varphi} - C_t - q_t.$$

By using (37), the equation of the NRE accumulation and that of pollution respectively becomes:

$$\dot{S}_t = -E_{st} = -\xi(E_{1t} + E_{2t}) + \xi(\theta_1(q_t) + \theta_2(q_t))$$

and

$$\dot{Z}_t = E_{st} = \xi(E_{1t} + E_{2t}) - \xi(\theta_1(q_t) + \theta_2(q_t)).$$

## 8.7 Appendix E

1. **Continuity of $\lambda_1$**

- At t = $T_1$

$$\underbrace{\{(\overline{K_1} - K_0) * \frac{\alpha_2(1-\phi)(1-\delta) - \rho}{\delta}\}^{-\delta} e^{(\rho - \alpha_2(1-\phi))T_1}}_{First \quad regime} = \underbrace{\{(\overline{K_2} - K_{T_1}) * \frac{\alpha_2(1-\phi)(1-\delta) - \rho}{\delta}\}^{-\delta}}_{Second\, regime}$$

This implies that :

$$K_{T_1} = -(\overline{K_1} - K_0) * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta} T_1} + \overline{K}_2 \tag{41}$$

- At t = $T_2$

$$\underbrace{\{(\overline{K_2}-K_{T_1})*\frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\}^{-\delta}*e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}(T_2-T_1)}}_{Second\ regime}=\underbrace{\{(-K_{T_2}*\frac{\alpha_2-\rho-\delta\Lambda}{\Theta\delta}\}^{-\delta}}_{Third\quad regime}$$

That leads to:

$$K_{T_2}=(K_0-\overline{K_1})*\frac{\Theta[\alpha_2(1-\phi)(1-\delta)-\rho]}{\alpha_2-\rho-\delta\Lambda}*e^{\frac{\alpha_2(1-\phi)-\rho}{\delta^2}[(1+\delta)T_1-T_2]} \tag{42}$$

**2. Continuity of $K$**

- At t = $T_1$

As the part $\theta$ of capital is lost from the first to the second regime, we have that: $\overline{K}=(1-\theta)\underline{K}$, with $\overline{K}$ the capital in the second regime and $\underline{K}$ the capital in the first regime.

K is continuous at $T_1$ so that:

$$\underbrace{-(\overline{K_2}-K_{T_1})+\overline{K_2}}_{Second\quad regime}=(1-\theta)\underbrace{[-(\overline{K_1}-K_0)e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})T_1}+\overline{K_1}]}_{First\ regime}$$

This gives:

$$K_{T_1}=(1-\theta)\left\{(K_0-\overline{K_1})*e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}+\overline{K_1}\right\} \tag{43}$$

As (41) and (43) are the same expression of $K_{T_1}$, we have the following equality:

(41)=(43), that implies:

$$-(\overline{K_1}-K_0)*e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}+\overline{K}_2=(1-\theta)\left\{(K_0-\overline{K_1})*e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}+\overline{K}_1\right\}$$

This helps to deduce the expression of $\overline{K}_2$ as a function of $\overline{K}_1$:

$$\overline{K}_2=\theta(\overline{K_1}-K_0)*e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}+(1-\theta)\overline{K_1} \tag{44}$$

- At t = $T_2$

From the second regime to the third, capital is not lost so that: $\overline{\overline{K}}=\overline{K}$, with $\overline{K}$ the capital in the second regime and $\overline{\overline{K}}$ the capital in the third regime.

Continuity of capital implies that:

$$\underbrace{-(\overline{K_2}-K_{T_1})*e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(T_2-T_1)}+\overline{K_2}}_{Second\quad regime}=\underbrace{-\frac{\Theta\delta}{\alpha_2-\rho-\delta\Lambda}\lambda_{T_2}^{-\frac{1}{\delta}}}_{Third\ regime}$$

Using continuity of $\lambda_1$ and (41) we have the following equality:

$$\lambda_{T_2}=\lambda_{1T_2}=\lambda_{1T_1}*e^{(\rho-\alpha_2(1-\phi))(T_2-T_1)}=\{(\overline{K_2}-K_{T_1})*\frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\}^{-\delta}*e^{(\rho-\alpha_2(1-\phi))(T_2-T_1)}$$

and $K_{T_1} - \overline{K}_2 = -(\overline{K_1} - K_0) * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}$.

From that, we get:

$(\overline{K_1}-K_0)*e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}*e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(T_2-T_1)}+\overline{K_2} = -\frac{\Theta\delta}{\alpha_2-\rho-\delta\Lambda}*(\overline{K_1}-K_0)*\frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}*$
$e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1} * e^{\frac{(\alpha_2(1-\phi)-\rho)}{\delta}(T_2-T_1)}$

That gives:

$$\overline{K}_2 = \left\{\frac{\Theta[\alpha_2(1-\phi)(1-\delta)-\rho]}{\alpha_2-\rho-\delta\Lambda}+1\right\}(K_0-\overline{K_1}) * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_2} \tag{45}$$

(44)=(45) leads :

$$\overline{K}_1 = \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0 \tag{46}$$

with $f(T_1,T_2) = \Gamma * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_2} - \theta e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}$ and $\Gamma = -1 - \frac{\Theta[\alpha_2(1-\phi)(1-\delta)-\rho]}{\alpha_2-\rho-\delta\Lambda}$

(46) in (45) leads: $\overline{K}_2 = -\Gamma\left\{K_0 - \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0\right\} * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_2}$

(43) becomes: $K_{T_1} = (1-\theta) * \left\{K_0 - \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0\right\} * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}$

(42) becomes: $K_{T_2} = \left\{K_0 - \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0\right\} * \frac{\Theta[\alpha_2(1-\phi)(1-\delta)-\rho]}{\alpha_2-\rho-\delta\Lambda} * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta^2}[(1+\delta)T_1-T_2]}$

and $\lambda_{T_2} = \{(\overline{K_1}-K_0) * e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1} * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\}^{-\delta} * e^{(\rho-\alpha_2(1-\phi))(T_2-T_1)}$

$= \left\{(-K_0 + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\right\}^{-\delta} * e^{(\rho-\alpha_2(1-\phi))T_2}$

Also, we have:

$\lambda_{2.0} = -\frac{1}{\xi}[(-\frac{\overline{Z}}{\xi}+\frac{\alpha_2(1-\phi)}{\beta_2}\frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0*T_1-\frac{\alpha_2(1-\phi)\delta}{\beta_2(\alpha_2(1-\phi)-\rho)}(\frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0-K_0)[e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})*T_1}-$
$1]) * \frac{\rho}{\delta[e^{-\frac{\rho}{\delta}T_1}-1]}]^{-\delta}$

and

$\lambda_{2.T_1} = \frac{1}{\xi}[(\frac{-S_0+\overline{Z}}{\xi}+\frac{\alpha_2(1-\phi)}{\beta_2}\overline{K_2}(T_2-T_1)-\frac{\alpha_2(1-\phi)\delta}{\beta_2(\alpha_2(1-\phi)-\rho)}(\overline{K_1}-K_0)e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1}*[e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}*(T_2-T_1)}-$
$1])\frac{\rho}{\delta[e^{-\frac{\rho}{\delta}T_1}-1]}]^{-\delta}$

## 3. Dynamics of variables

We use the above results to get an expression for state and control variables.

- First regime

$K_t = \left\{K_0 - \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0\right\} * e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t} + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}$

$\lambda_1 = \left\{(-K_0 + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\right\}^{-\delta} * e^{(\rho-\alpha_2(1-\phi))t}$

$\lambda_2 = \lambda_{2.0}e^{\rho t}$

$C_t = \left\{(-K_0 + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\right\} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t}$

$E_{2t} = (-\lambda_{2.0}\xi)^{-\frac{1}{\delta}} e^{-\frac{\rho}{\delta}t}$

$E_{1t} = \frac{\alpha_2(1-\phi)}{\beta_2}[\left\{K_0 - \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0\right\} * e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t} + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0]$

- Second regime

$\lambda_{1T_1} = \left\{(-K_0 + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\right\}^{-\delta} * e^{(\rho-\alpha_2(1-\phi))T_1}$

$\lambda_1 = \left\{(-K_0 + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\right\}^{-\delta} * e^{(\rho-\alpha_2(1-\phi))t}$

$\lambda_2 = \lambda_{2.T_1} e^{\rho(t-T_1)}$

$K_t = -(\overline{K_1} - K_0) * e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})t} + \overline{K_2}$

$E_{1t} = \frac{\alpha_2(1-\phi)}{\beta_2}\overline{K}_t$

$C_t = \lambda_{1.T_1}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2(1-\phi)-\rho}{\delta})(t-T_1)}$

$E_{2t} = (\lambda_{2.T_1}\xi)^{-\frac{1}{\delta}} e^{\frac{-\rho}{\delta}(t-T_1)}$

- Third regime

$\lambda_{T_2} = \left\{(-K_0 + \frac{f(T_1,T_2)}{f(T_1,T_2)+\theta-1}K_0) * \frac{\alpha_2(1-\phi)(1-\delta)-\rho}{\delta}\right\}^{-\delta} * e^{(\rho-\alpha_2(1-\phi))T_2}$

$K_t = -\frac{\Theta\delta}{\alpha_2-\rho-\delta\Lambda}\lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2-\rho}{\delta})(t-T_2)}$

$\lambda_t = \lambda_{T_2} e^{(\rho-\alpha_2)(t-T_2)}$

$C_t = \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2-\rho}{\delta})(t-T_2)}$

$E_{2t} = (\frac{\alpha_2}{\varphi})^{-\frac{1}{\delta}} \lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2-\rho}{\delta})(t-T_2)}$

$E_{1t} = \frac{\alpha_2}{\alpha_2+\beta_2\varphi}(\varphi\overline{\overline{K}}_t - E_{2t})$

4. Equality of Hamiltonian

The last optimality condition is the equality of Hamiltonian at the switching time $T_1$ and $T_2$.

- At t = $T_1$: the equality is between the first and the second regime.

$H_1(T_1^*) = H_2(T_1^*)$

$$\Rightarrow \underbrace{\frac{E_2^{1-\delta}}{1-\delta} + \lambda_1\alpha_2 K + \lambda_2\xi(E_1 + E_2)}_{First\ regime} = \underbrace{\frac{E_2^{1-\delta}}{1-\delta} + \lambda_1\alpha_2 K - \lambda_2\xi(E_1 + E_2)}_{second\ regime} \quad (47)$$

- At t = $T_2$: the equality is between the second and the third regime.

  $H_2(T_2^*) = H_3(T_2^*)$

$$\Rightarrow \underbrace{\frac{E_2^{1-\delta}}{1-\delta} + \lambda_1\alpha_2 K - \lambda_2\xi(E_1+E_2)}_{Second\ regime} = \underbrace{\frac{E_2^{1-\delta}}{1-\delta} + \lambda[\alpha_2(K - \frac{1}{\varphi}E_1 - \frac{1}{\varphi}E_2)]}_{Third\ regime} \qquad (48)$$

Solving simultaneously and numerically (47) and (48), we can get $T_1$ and $T_2$. Now, let look at corners solutions before providing the value function given by the numerical results of the switching time.

## 8.8 Appendix F

### 8.8.1 Third regime

The Hamiltonian is the following:

$$H_3 = \frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta} + \lambda[\alpha_2 K_t - \alpha_2\frac{(E_{1t}+E_{2t}) - (\vartheta_1(q_t)+\vartheta_2(q_t))}{\varphi} - C_t - q_t].$$

All the previous FOCs remain the same. The main change comes from the FOC with respect to the investment $q_t$:

$$\vartheta_1^{'}(q_t) + \vartheta_2^{'}(q_t) = \frac{\varphi}{\alpha_2} \qquad (49)$$

The equation (49) highlights the arbitrage condition between the reduction of resources as a gain from the energy savings technologies and the constant marginal cost of investment. The solution of (49) gives the optimal investment in energy saving technologies. Now, let us specify the energy saving $\vartheta_i(q_t)$ as a class of power function $cq_t^{\sigma_i}$ where $i \in \{1,2\}$ and c, $\sigma_i$ are the parameters. Moreover, we set $c = 1$ and $\sigma_i \in [0,1]$ in order to meet the required properties defined before.

Without loss of generality, let us assume that investment in EST yields the same productivity either at household level or at industry level such that $\sigma_1 = \sigma_2 = \sigma$. Thus, we get that:

$$q^* = [\frac{\varphi}{2\sigma\alpha_2}]^{\frac{1}{\sigma-1}} \qquad (50)$$

By replacing the optimal value of the investment (50) into the equation of capital accumulation, we can solve the model as before to get the following expression of capital.

$$K_t = -\frac{\Theta\delta}{\alpha_2 - \rho - \delta\Lambda}\lambda_{T_2}^{-\frac{1}{\delta}} e^{(\frac{\alpha_2-\rho}{\delta})(t-T_2)} - \frac{\varpi}{\Lambda},$$

where $\varpi = \frac{2\Lambda}{\varphi}(\frac{\varphi}{2\sigma\alpha_2})^{\frac{\sigma}{\sigma-1}} - (\frac{\varphi}{2\sigma\alpha_2})^{\frac{1}{\sigma-1}}$ and the others are the same as defined before.

In this new expression of capital, we have an additional component $-\frac{\varpi}{\Lambda}$ due to the investment in EST. This additional component is negative in the sense that it negatively affects the level of capital. In fact, the economy additionally uses a part of income to invest in EST. This part could have been invested in productive sector (final goods and energy) or consumed by households. Hence the share of the income that goes to investment is reduced.

### 8.8.2 Second regime

The Hamiltonian is the following:

$$H_2 = \frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta} + \lambda_1[\alpha_2(1-\phi)K - C_t - q_t] - \lambda_2\xi[(E_{1t}+E_{2t}) - (\vartheta_1(q_t) + \vartheta_2(q_t))].$$

As before, the only change is the FOC with respect to $q_t$:

$$\vartheta_1'(q_t) + \vartheta_2'(q_t) = \frac{\lambda_1}{\xi\lambda_2} \tag{51}$$

Using the same specifications as before, the solution of (51) is :

$$q^* = [\frac{\lambda_1}{2\sigma\xi\lambda_2}]^{\frac{1}{\sigma-1}} \tag{52}$$

The equation (52) helps to solve the model during the second regime as before. The expression of the capital during the second regime becomes:

$$K_t - \overline{K_2} = -\lambda_{1.T_1}^{-\frac{1}{\delta}} * \frac{\delta}{\alpha_2(1-\phi)(1-\delta)-\rho} e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}(t-T_1)} + [\frac{\lambda_{1.T_1}}{2\sigma\xi\lambda_{2.T_1}}]^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\alpha_2(1-\phi)\sigma} e^{-\frac{\alpha_2(1-\phi)}{\sigma-1}(t-T_1)}.$$

We should also notice here that the level of capital at each period of time during the second regime has a second negative component. As the share of the income that goes to investment is reduced by investment in energy saving technologies, one should expect a decrease in capital.

As we did in the case without any investments in EST, all the NRE are extracted during the first and the second regimes such that:

$$S_0 - \overline{Z} = \int_{T_1}^{T_2} \xi(E_{1t} + E_{2t} - \vartheta_1 - \vartheta_2)dt,$$

By solving the above equation, we get that:

$$S_0 - \overline{Z} = \xi\frac{\alpha_2}{\beta_2}\overline{K_2}\,[T_2 - T_1] + H_0\left[e^{-\frac{\rho}{\delta}(T_2-T_1)} - 1\right]\lambda_{2.T_1}^{-\frac{1}{\delta}} + H_1\left[e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}(T_2-T_1)} - 1\right]\lambda_{1.T_1}^{-\frac{1}{\delta}} + H_2\left[e^{\frac{-\alpha_2(1-\phi)}{\sigma-1}(T_2-T_1)} - 1\right][\frac{\lambda_{1.T_1}}{\lambda_{2.T_1}}]^{\frac{1}{\sigma-1}} + H_3\left[e^{\frac{-\alpha_2\sigma(1-\phi)}{\sigma-1}(T_2-T_1)} - 1\right][\frac{\lambda_{1.T_1}}{\lambda_{2.T_1}}]^{\frac{\sigma}{\sigma-1}} \quad \textbf{(Eq A)}$$

where $H_0 = -\xi^{\frac{\delta-1}{\delta}}\frac{\delta}{\rho}$, $H_1 = -\frac{\xi\delta^2\alpha_2}{\beta_2[\alpha_2(1-\phi)(1-\delta)-\rho][\alpha_2(1-\phi)-\rho]}$, $H_2 = -\frac{\xi(\sigma-1)^2\alpha_2}{\beta_2(2\sigma\xi)^{\frac{1}{\sigma-1}}\sigma\alpha_2^2(1-\phi)^2}$ and $H_3 = \frac{2\xi(\sigma-1)}{\alpha_2(1-\phi)\sigma(2\sigma\xi)^{\frac{\sigma}{\sigma-1}}}$

### 8.8.3 First regime

The Hamiltonian is the following:

$$H_1 = \frac{C^{1-\delta}}{1-\delta} + \frac{E_2^{1-\delta}}{1-\delta} + \lambda_1(\alpha_2(1-\phi)K - C_t - q_t) + \lambda_2\xi[(E_{1t}+E_{2t}) - (\vartheta_1(q_t)+\vartheta_2(q_t))]$$

Like in the second regime the optimal investment in EST is:

$$q^* = [-\frac{\lambda_1}{2\sigma\xi\lambda_2}]^{\frac{1}{\sigma-1}} \tag{53}$$

We then solve the equation of capital accumulation to get the following expression of capital during the first regime.

$$K_t - \overline{K_1} = -\lambda_{1.0}^{-\frac{1}{\delta}} \frac{\delta}{\alpha_2(1-\phi)(1-\delta)-\rho} e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}t} + [-\frac{\lambda_{1.0}}{2\sigma\xi\lambda_{2.0}}]^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma\alpha_2(1-\phi)} e^{\frac{-\alpha_2(1-\phi)}{\sigma-1}t}$$

We still have an additional negative component of the capital due to investment in energy saving technologies.

At the end of the first regime, we cross the pollution threshold so that:

$$\overline{Z} = \int_0^{T_1} \xi(E_{1t}+E_{2t}-2\vartheta_t^*)dt$$

By solving the above equation as before, we get the following expression:

$$\overline{Z} = \xi\frac{\alpha_2}{\beta_2}\overline{K_1}T_1 + H_0\left[e^{-\frac{\rho}{\delta}T_1} - 1\right](-\lambda_{2.0})^{-\frac{1}{\delta}} + H_1\left[e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1} - 1\right]\lambda_{1.0}^{-\frac{1}{\delta}} + H_4\left[e^{\frac{-\alpha_2(1-\phi)}{\sigma-1}T_1} - 1\right]\lambda_{1.0}^{\frac{1}{-\delta}} + H_3\left[e^{\frac{-\alpha_2\sigma(1-\phi)}{\sigma-1}T_1} - 1\right]\left[\frac{\lambda_{1.0}}{\lambda_{2.0}}\right]^{\frac{\sigma}{\sigma-1}},\ \textbf{(Eq B)}$$

where $H_0$, $H_1$ and $H_3$ are the same as defined before and $H_4 = \frac{\xi\delta(\sigma-1)\alpha_2}{\beta_2\alpha_2((1-\phi)[\alpha_2(1-\phi)(1-\delta)-\rho]}$.

### 8.8.4 Boundary conditions

Like in the case without any investments in EST, we use some boundary conditions as in Boucekkine et alii (2013). Continuity of $\lambda_1$ and continuity of $K$ at the switching times $T_1$ and $T_2$ give the following equation.

$$H_5\lambda_{1.0}^{-\frac{1}{\delta}} - \frac{\varpi}{\Lambda} = H_6\lambda_{1.0}^{-\frac{1}{\delta}} + H_7\left[\frac{\lambda_{1.0}}{2\sigma\xi\lambda_{2.T_1}}\right]^{\frac{1}{\sigma-1}} + H_8\left[\frac{-\lambda_{1.0}}{2\sigma\xi\lambda_{2.0}}\right]^{\frac{1}{\sigma-1}} + (1-\theta)K_0 \quad \textbf{(Eq C)}$$

Where $H_5 = -\frac{\Theta\delta}{\alpha_2-\rho-\delta\Lambda}e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_2}$, $H_6 = \frac{\delta}{\alpha_2(1-\phi)(1-\delta)-\rho}\left[\theta e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_1} + (1-\phi) - e^{\frac{\alpha_2(1-\phi)-\rho}{\delta}T_2}\right]$,

$$H_7 = \frac{\sigma-1}{\alpha_2\sigma(1-\phi)}\left[-e^{\frac{\rho-\alpha_2(1-\phi)}{\sigma-1}T_1} + e^{[\frac{\rho}{\sigma-1}T_1 - \frac{\alpha_2(1-\phi)}{\sigma-1}T_2]}\right] \text{ and } H_8 = \frac{(\sigma-1)(1-\theta)}{\alpha_2\sigma(1-\phi)}\left[-1 + e^{\frac{-\alpha_2(1-\phi)}{\sigma-1}T_1}\right].$$

Eq A, Eq B and Eq C express three different relationships between $\lambda_{1.0}$, $\lambda_{2.0}$ and $\lambda_{2.T_1}$ that we can simultaneously solve. Additionally, we simultaneously and numerically solve the equality of Hamiltonians at the switching time $T_1$ and $T_2$ to get $T_1$ and $T_2$.